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Linearizability of linear systems perturbed by fifth degree homogeneous polynomials

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Abstract
We present the necessary and sufficient conditions for linearizability of the planar complex system $\dot{x} = x + P(x, y)$, $\dot{y} = -y + Q(x, y)$, where P and Q are homogeneous polynomials of degree 5. Using these conditions, we also give the complete solution for the isochronicity of real systems in the form of linear oscillator perturbed by fifth degree homogeneous polynomials.

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1. Introduction

Consider a planar autonomous analytical differential system in the form of a linear centre perturbed by higher order terms, that is,

$$\dot{u} = -v + \sum_{i+j=2}^{\infty} \alpha_{ij} u^i v^j = -v + U(u, v), \qquad \dot{v} = u + \sum_{i+j=2}^{\infty} \beta_{ij} u^i v^j = u + V(u, v), \tag{1}$$

where U and V are real analytic functions whose series expansions in a neighbourhood of the origin start with terms of the second degree or higher. Conversion to polar coordinates shows that near the origin either all non-stationary trajectories of (1) are ovals (in which case the origin is called a *centre*) or they are all spirals (in which case the origin is called a *focus*). If all solutions near $u = 0$, $v = 0$ are periodic (that is, the origin is a centre), the problem then arises to determine whether the period of oscillations is constant for all solutions near the origin. A centre with such property is called the *isochronous* centre. It follows from a result of Poincaré and Lyapunov that the centre of (1) is isochronous if and only if it is *linearizable*, that is, if there exists an analytic transformation $X = u + \sum_{i+j=2}^{\infty} d_{ij} u^i v^j$, $Y = v + \sum_{i+j=2}^{\infty} s_{ij} u^i v^j$, which brings (1) into the linear system $\dot{X} = -Y$, $\dot{Y} = X$.

Although the study of isochronous oscillations goes back at least to Huygens who investigated the motion of a cycloidal pendulum, at present the problem is of renewed interest. Starting from the 1960s many studies have been devoted to the investigation of the isochronicity and linearizability problems for various subfamilies of system (1). In 1964, Loud [16] classified isochronous centres of system (1) with U and V being homogeneous polynomials of degree 2, and in 1969, Pleshkan [19] found all isochronous centres in the family (1), where U and V are homogeneous polynomials of degree 3.

However, the classifications of isochronous centres in the form of linear centre perturbed by homogeneous polynomials of the fourth and fifth degrees turned out to be much more difficult. Up to now only partial results have been obtained. In particular, Chavarriga, Giné and García found isochronous centres for time-reversible systems (1) in the case of perturbation of the fourth and fifth degrees [4, 5] (by definition, system (1) is time reversible if it is invariant under reflection with respect to a line passing through the origin and a change in the direction of time).

As we have mentioned above, the problem of isochronicity is equivalent to the problem of linearizability. To study the linearizability, it is convenient to introduce a complex structure on the phase plane (u, v) by setting $x = u + \mathrm{i}v$. Then we obtain from system (1) the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \widetilde{P}(x, \bar{x}). \tag{2}$$

Adjoining to the latter equation its complex conjugate we have the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \widetilde{P}(x, \bar{x}), \qquad \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \overline{\widetilde{P}(x, \bar{x})}.$$

Let us consider $\bar{x}$ as a new variable y and $\overline{\widetilde{P}}$ as a new function Q. Then, in the case when U and V are polynomials of degree n, from the latter system we obtain a system of two complex differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i}\left(x - \sum_{p+q=1}^{n-1} a_{p,q} x^{p+1} y^q\right), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\mathrm{i}\left(y - \sum_{p+q=1}^{n-1} b_{q,p} x^q y^{p+1}\right), \tag{3}$$

here and in similar systems below $p \geqslant -1, q \geqslant 0$. We denote the vector of coefficients of system (3) by (A, B), that is, $(A, B) = (a_{1,0}, a_{0,1}, \ldots, a_{-1,n}, b_{1,0}, b_{0,1}, \ldots, b_{n,-1})$. We use the same notation also for the vector of coefficients of system (5).

The linearizability problem for system (3) is the problem to decide whether the system can be transformed to the linear system $\dot{z}_1 = \mathrm{i}z_1$, $\dot{z}_2 = -\mathrm{i}z_2$ by means of a formal change of the phase variables

$$\begin{aligned} z_1 &= x + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(A, B) x^m y^j, \\ z_2 &= y + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(A, B) x^m y^j. \end{aligned} \tag{4}$$

If such transformation exists we say that the system is *linearizable* (it is also said that there is a *linearizable centre* at the origin). It is well known (see, e.g., [3]) that if there exist formal series (4) linearizing (3) then the series converge in a neighbourhood of the origin.

After the change of time i$\mathrm{d}t = \mathrm{d}\tau$ and then rewriting t instead of τ we obtain from system (3) the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \sum_{p+q=1}^{n-1} a_{p,q} x^{p+1} y^q = x + P(x, y) = \widetilde{P}(x, y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y + \sum_{p+q=1}^{n-1} b_{q,p} x^q y^{p+1} = -y + Q(x, y) = \widetilde{Q}(x, y). \tag{5}$$

It is clear that the conditions for linearizability of system (5), that is, the condition under which, by (4), the system can be transformed to the linear system

$$\dot{z}_1 = z_1, \qquad \dot{z}_2 = -z_2, \tag{6}$$

are the same as the conditions for linearizability of system (3), so we will study system (5). The problem of linearizability for systems (5) is a generalization of the problem of linearizability (isochronicity) for polynomial systems (1) in the sense that if we know all linearizable systems within a given family (5) then going back to (3) and then to the real coordinates u, v we obtain all linearizable systems in the corresponding real family (1). However, there are linearizable systems (3) which do not have a real 'preimage' (counterpart).

The linearizability problem for system (5) with both P and Q being homogeneous polynomials of degrees 2 and 3 has been solved in [9]; some particular families of linearizable cubic systems were presented in [21]. Of course, we have mentioned here only very few contributions to the problem of isochronicity and linearizability. For more references, the interested reader can consult, e.g., [2, 5, 6, 8, 22].

In this paper, we present the conditions for linearizability of system (5) with P and Q being homogeneous polynomials of the fifth degree (system (21)). For this system, the nonzero linearizability quantities $i_2, i_4, i_6, \ldots$ (see definition in section 2) are polynomials of degrees $1, 2, 3, \ldots,$ respectively, whereas for the case of system (5) where P and Q are homogeneous polynomials of the fourth degree the nonzero quantities $i_3, i_6, i_9, \ldots$ are polynomials of degrees $2, 4, 6, \ldots,$ respectively. Thus, in the quartic case we have to deal with polynomials of higher degrees than in the quintic one and, therefore, it is much easier to find the irreducible decomposition of the variety[5] defined by the linearizability quantities in the case when P and Q are homogeneous quintic nonlinearities than in the case when they are homogeneous quartic nonlinearities. In the last section, we compare the obtained results with those of [5].

2. Preliminaries

In this section, we briefly describe a general approach to studying the linearizability problem for polynomial systems (5).

The first step is the calculation of the so-called linearizability quantities, which are polynomials of the coefficients $a_{k,p}$, $b_{p,k}$ of system (5). Taking derivatives with respect to t in both parts of each of the equalities in (4), we obtain

$$\dot{z}_1 = \dot{x} + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j} (m x^{m-1} y^j \dot{x} + j x^m y^{j-1} \dot{y}),$$

[5] We recall that the variety of a given polynomial ideal $F = \langle f_1, \ldots, f_s \rangle$ is the set of common zeros of polynomials $f_1, \ldots, f_s$; it is denoted by $V(F)$.

$$\dot{z}_2 = \dot{y} + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(mx^{m-1}y^j\dot{x} + jx^m y^{j-1}\dot{y}).$$

Equating coefficients of the terms $x^{q_1+1}y^{q_2}$, $x^{q_1}y^{q_2+1}$, we obtain the recurrence formulae

$$(q_1 - q_2)u^{(1)}_{q_1,q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1} \left[(s_1+1)u^{(1)}_{s_1,s_2}a_{q_1-s_1,q_2-s_2} - s_2u^{(1)}_{s_1,s_2}b_{q_1-s_1,q_2-s_2}\right], \quad (7)$$

$$(q_1 - q_2)u^{(2)}_{q_1,q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1} \left[s_1u^{(2)}_{s_1,s_2}a_{q_1-s_1,q_2-s_2} - (s_2+1)u^{(2)}_{s_1,s_2}b_{q_1-s_1,q_2-s_2}\right], \quad (8)$$

where $s_1, s_2 \geqslant -1, q_1, q_2 \geqslant -1, q_1 + q_2 \geqslant 0, u^{(1)}_{1,-1} = u^{(1)}_{-1,1} = 0, u^{(2)}_{1,-1} = u^{(2)}_{-1,1} = 0, u^{(1)}_{0,0} = u^{(2)}_{0,0} = 1$, and we set $a_{q,m} = b_{m,q} = 0$, if $q + m < 1$.

Thus, we see that the coefficients $u^{(1)}_{q_1,q_2}, u^{(2)}_{q_1,q_2}$ of transformation (4) can be computed step by step using formulae (7) and (8). In the case $q_1 = q_2 = q$ the coefficients $u^{(1)}_{q,q}, u^{(2)}_{q,q}$ can be chosen arbitrary (we set $u^{(1)}_{q,q} = u^{(2)}_{q,q} = 0$). The system is linearizable if and only if the quantities on the right-hand side of (7) and (8) are equal to zero for all $q_1 = q_2 = q \in \mathbb{N}$. As a matter of definition, in the case $q_1 = q_2 = q$ we denote the polynomials on the right-hand side of (7) by i_q and on the right-hand side of (8) by $-j_q$ and call them *qth linearizability quantities*. We see that system (5) with the given coefficients (A, B) is linearizable if and only if $i_k(A, B) = j_k(A, B) = 0$ for all $k \in \mathbb{N}$.

In the space of the parameters of a given family of systems (5) the set of all linearizable systems is an affine variety V of the ideal $\langle i_1, j_1, i_2, j_2, \ldots\rangle$. Due to the Hilbert basis theorem there exists $N \in \mathbb{N}$ such that V is equal to the variety of the ideal $\langle i_1, j_1, \ldots i_N, j_N\rangle$; however, the theorem does not give any idea how to find the number N. A practical way to compute V is to take N_0 equal to a half of the number of parameters of the system, to compute the ideal $I_{N_0} = \langle i_1, j_1, \ldots, i_{N_0}, j_{N_0}\rangle$ and to find the minimal associate primes of the ideal I_{N_0}, which define the irreducible decomposition of the variety V_{N_0} of the ideal I_{N_0}, $V_{N_0} = V_1 \cup \cdots \cup V_s$. Then for each component V_k $(k = 1, \ldots, s)$ one tries to find linearizing substitutions for all systems from the component (or at least to prove the existence of such substitutions). If for all systems from V_{N_0} linearizations exist then $V = V_{N_0}$ and the problem is solved.

The most powerful method to find a linearizing substitution is the so-called Darboux linearization. By definition, a Darboux linearization [17] of system (5) is a change of variables

$$z_1 = H_1(x, y), \qquad z_2 = H_2(x, y), \quad (9)$$

which transforms the system to the linear system (6), and such that at least one of the functions H_1, H_2 is of the form $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ (α_j's are complex numbers), where $f_i(x, y)$'s are some functions (called Darboux functions) satisfying the equation

$$\frac{\partial f_i}{\partial x}\widetilde{P} + \frac{\partial f_i}{\partial y}\widetilde{Q} = K_i f_i,$$

with K_i's being some polynomials. The polynomial $K_i(x, y)$ is called the cofactor of $f_i(x, y)$. A simple computation shows that if there are Darboux functions $f_1, f_2, \ldots, f_k$ with the cofactors $K_1, K_2, \ldots, K_k$ satisfying

$$\sum_{i=1}^{k} \alpha_i K_i = 0, \quad (10)$$

then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ is a first integral of system (5), and if

$$\sum_{i=1}^{k} \alpha_i K_i + \widetilde{P}'_x + \widetilde{Q}'_y = 0, \tag{11}$$

then the equation admits the integrating factor $\mu = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$.

Similarly, one can construct Darboux linearizations. In particular, assume that system (5) has a first integral of the form

$$\Phi(x, y) = xy \left(1 + \sum_{k+j=1}^{\infty} v_{k,j} x^k y^j\right) \tag{12}$$

and the Darboux functions $f_1, f_2, \ldots, f_s$, which are analytic functions, such that for all $m = 1, \ldots, s$, $f_m(0, 0) = 1$. Let $K_1, \ldots, K_s$ be the corresponding cofactors. In such a case, if

$$(1 - c)\frac{P}{x} - c\frac{Q}{y} + \sum_{j=1}^{s} \alpha_j K_j = 1 \tag{13}$$

then the first equation of (5) is linearized by the substitution

$$z_1 = x^{1-c} y^{-c} \Phi^c f_1^{\alpha_1} f_2^{\alpha_2} \dot{s} f_s^{\alpha_s}, \tag{14}$$

and if

$$-c\frac{P}{x} + (1 - c)\frac{Q}{y} + \sum_{j=1}^{s} \alpha_j K_j = -1 \tag{15}$$

then the second equation of (5) is linearizable by the substitution

$$z_2 = x^{-c} y^{1-c} \Phi^c f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_s^{\alpha_s}. \tag{16}$$

For most cases studied in this work it is possible to find Darboux linearizations of the form (14) or (16) with $c = 0$; only for case 5 of theorem 1, we have used the integral $\Phi(x, y)$ in order to obtain the linearization of the form (14) and (16) with $c \neq 0$.

If system (5) is such that only one of conditions (14), (16) is satisfied, let us say (16), but it has a first integral $\Psi(x, y)$ of the form

$$\Psi(x, y) = xy + \sum_{l+j=3}^{\infty} v_{l,j} x^l y^j, \tag{17}$$

which we call the *Lyapunov first integral*, then (5) is linearizable by the change

$$z_1 = \Psi(x, y)/H_2(x, y), \qquad z_2 = H_2(x, y). \tag{18}$$

Sometimes the following observation is helpful. Assuming that (5) has a first integral (17), we can write the system in the form

$$\dot{x} = r\Psi_y, \qquad \dot{y} = -r\Psi_x \tag{19}$$

for some analytic function $r(x, y)$ with $r(0, 0) = 1$. Eliminating Ψ in (19) gives us

$$\dot{r} = \operatorname{div}(\dot{x}, \dot{y})r, \tag{20}$$

which means that r is a Darboux function with the cofactor $\operatorname{div}(\dot{x}, \dot{y})$.

More details on the Darboux method of integration and linearization can be found in [8, 9, 17].

3. The conditions for linearizability

In this section, we will find the conditions for linearizability of system (5), where P and Q are homogeneous polynomials of degree 5, that is,

$$\begin{aligned}\dot{x} &= x - a_1x^5 - a_2x^4y - a_3x^3y^2 - a_4x^2y^3 - a_5xy^4 - a_6y^5,\\ \dot{y} &= -y + b_6x^5 + b_5x^4y + b_4x^3y^2 + b_3x^2y^3 + b_2xy^4 + b_1y^5.\end{aligned} \tag{21}$$

Note that for (21) $i_2 = a_3$, $j_2 = b_3$; therefore if $|a_3| + |b_3| \neq 0$, then the system is not linearizable, so from now on we assume that in (21) $a_3 = b_3 = 0$.

For system (21) (with $a_3 = b_3 = 0$) using a straightforward modification of Mathematica code from [21, appendix], we have computed the first six different from zero linearizability quantities $i_4, j_4, i_6, j_6, \ldots, i_{14}, j_{14}$ (the quantities of system (21) with sub-indexes difference from $2k$ are equal to zero). The polynomials are too long, so we do not present them here; however, the interested reader can easily compute them by using any available computer algebra systems by algorithms from [9] or [21], for instance. To find the necessary conditions for linearizability of system (21) it is sufficient to find the irreducible decomposition of the variety of the ideal $I = \langle i_4, j_4, \ldots, i_{14}, j_{14}\rangle$. To do so, we used the routine *minAssChar* [18] of *Singular* [14] which finds the minimal associate primes of a polynomial ideal by means of the characteristic sets method [23]. Note that if for system (21) $a_6 \neq 0$, $b_6 \neq 0$, then by a linear transformation we can set in (21) $a_6 = b_6 = 1$. Since the ideal is huge and the calculations are tremendous, in order to be able to carry out them using the above observation we split our system (21) into three systems considering separately the cases:

$$(\alpha)\ a_6 = b_6 = 0, \qquad (\beta)\ a_6 = 1, b_6 = 0, \qquad (\gamma)\ a_6 = b_6 = 1.$$

For cases (β) and (γ) at our computational facilities the decomposition is still impossible in the rational arithmetic; however, we have succeeded to find it by computing in the field of characteristic 32 003 and then, using the reconstruction to rational arithmetic[6], we have obtained the necessary conditions for linearizability presented in theorems 1 and 3. In the proofs of the theorems, we show that these conditions are also the sufficient conditions for linearizability of the corresponding systems.

To perform the rational reconstruction, that is, to reconstruct $p/s \in \mathbb{Q}$ given its image $t \in \mathbb{Z}/m$, we use the following algorithm [24, 25] (in the algorithm $\lfloor\cdot\rfloor$ stands for the floor function).

Algorithm RATCONVERT(c, m)

(1) $u = (u_1, u_2, u_3) := (1, 0, m)$, $v = (v_1, v_2, v_3) := (0, 1, c)$
(2) while $\sqrt{m/2} \leqslant v_3$ do $\{q := \lfloor u_3/v_3 \rfloor, r := u - qv, u := v, v := r\}$
(3) if $|v_2| \geqslant \sqrt{m/2}$ then error()
(4) return v_3, v_2

Given an integer number c and a natural number m, the algorithm produces integers v_3 and v_2 such that $v_3/v_2 \equiv c \bmod m$ and $|v_2|, |v_3| \leqslant \sqrt{m/2}$. Such a number v_3/v_2 need not exist. If it is the case, then the algorithm returns 'error()'.

We mention one more algorithm called *the radical membership test* (see, e.g., [10]), which was helpful for our calculations. Namely, given a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ and an ideal $J = \langle f_1, \ldots, f_s\rangle \subset \mathbb{C}[x_1, \ldots, x_n]$, f vanishes on the variety $V(J)$ of the ideal J if and only if the reduced Groebner basis of $\langle f_1, \ldots, f_s, 1 - wf\rangle \subset \mathbb{C}[w, x_1, \ldots, x_n]$ is $\{1\}$.

[6] To our knowledge modular arithmetic for the first time was used for computing normal forms of ODEs and for studies of the centre problem in [11].

Note that if we apply to the conditions of theorem 3 (which gives the conditions for the case (β)) the involution

$$a_i \leftrightarrow b_i, \tag{22}$$

then we obtain the conditions for linearizability of system (21) with $b_6 = 1, a_6 = 0$. Thus, theorems 1–3 provide the complete solution to the problem of linearizability for system (21).

Theorem 1. *System (21) with $a_6 = b_6 = 1$ is linearizable if and only if $a_3 = b_3 = 0$ and one of the following conditions holds:*

(1) $12b_1 - 5 = 6b_2 + 7 = 6b_4 - 1 = 12b_5 + 5 = 12a_5 + 5 = 6a_4 - 1 = 6a_2 + 7 = 12a_1 - 5 = 0$,
(2) $12b_1 + 5 = 6b_2 + 7 = 6b_4 - 1 = 12b_5 - 5 = 12a_5 - 5 = 6a_4 - 1 = 6a_2 + 7 = 12a_1 + 5 = 0$,
(3) $30b_2 - 84b_1 - 35 = 42b_4 - 6b_2 + 7 = 14b_5 - 5b_2 = 84a_5 + 30b_2 - 35 = 7a_4 + b_2 = 6a_2 + 6b_2 - 7 = 14a_1 + 5b_2 = 144b_1^2 + 60b_1 + 25 = 0$,
(4) $30b_2 + 84b_1 - 35 = 42b_4 - 6b_2 + 7 = 14b_5 + 5b_2 = 84a_5 - 30b_2 + 35 = 7a_4 + b_2 = 6a_2 + 6b_2 - 7 = 14a_1 - 5b_2 = 144b_1^2 - 60b_1 + 25 = 0$,
(5) $12b_1 - 17 = 2b_2 - 5 = 6b_4 + 5 = 4b_5 - 5 = 4a_5 - 5 = 6a_4 + 5 = 2a_2 - 5 = 12a_1 - 17 = 0$,
(6) $12b_1 + 17 = 2b_2 - 5 = 6b_4 + 5 = 4b_5 + 5 = 4a_5 + 5 = 6a_4 + 5 = 2a_2 - 5 = 12a_1 + 17 = 0$,
(7) $34b_2 - 60b_1 + 85 = 6b_4 - 2b_2 - 5 = 2b_5 + b_2 = 4a_5 - 2b_2 - 5 = 3a_4 + b_2 = 2a_2 + 2b_2 + 5 = 30a_1 + 17b_2 = 144b_1^2 - 204b_1 + 289 = 0$,
(8) $34b_2 + 60b_1 + 85 = 6b_4 - 2b_2 - 5 = 2b_5 - b_2 = 4a_5 + 2b_2 + 5 = 3a_4 + b_2 = 2a_2 + 2b_2 + 5 = 30a_1 - 17b_2 = 144b_1^2 + 204b_1 + 289 = 0$.

Proof 1. (i) To obtain the necessary condition for linearizability of system (21) with $a_6 = b_6 = 1$ we look for the irreducible decomposition of the variety $V(I)$ of the ideal $I = \langle i_4, j_4, \ldots, i_{14}, j_{14}, a_6 - 1, b_6 - 1\rangle$. To perform the decomposition we need to find the minimal associate primes of *I*. It can be done using, for instance, the routines *minAssChar* or *minAssGTZ* of *Singular* (we found that *minAssChar* is more efficient and used it for our calculations). Theoretically, both routines should return the minimal associate primes of a given polynomial ideal; however, the involved calculations are usually very heavy, so in practice the decomposition over $\mathbb{Q}$ is possible only for relatively simple polynomial ideals. In our case, we failed to perform the decomposition of *I* at the available computational facilities working over the field of rational numbers, but we have succeeded to find the minimal associate primes of *I* computing over the field of characteristic $m = 32\,003$. It should be noted that a result obtained using modular calculations not necessarily can be reconstructed to the true result (one reason for this is that, as it is indicated above, the reconstruction does not always succeed; another reason is that even an univariant polynomial irreducible over $\mathbb{Q}$ can be factorizable over $\mathbb{Z}/m$). In our case, the *minAssChar* returns the list of eight ideals. Six of them after the reconstruction using RATCONVERT give the varieties 1–6 of the statement of the theorem. However, the reconstruction of the remaining two ideals gives the ideals different from 7 and 8. Namely, one of these two ideals returning by *minAssChar* is $\widetilde{G} = \langle b_1^2 + 13336b_1 + 8225, b_2 + 15062b_1 - 15999, b_4 - 5647b_1, b_5 + 7531b_1 + 8002, a_5 - 7531b_1, a_4 + 5647b_1 + 5333, a_2 - 15062b_1, a_1 + b_1 + 13336\rangle$. Applying RATCONVERT we obtain from $\tilde{G}$ the ideal $G = \langle b_1^2 + 17b_1/12 - 15/214, b_2 + 30b_1/17 + 5/2, b_4 + 10b_1/17, a_5 - 15b_1/17, b_5 + 15b_1/17 + 5/4, a_4 - 10b_1/17 - 5/6, a_2 - 30b_1/17, a_1 + b_1 + 17/12\rangle$. Simple calculations show that not all polynomials of *I* vanish on the variety of $V(G)$, that is, $V(G)$ is not a component of $V(I)$. An empirical observation is that the simpler a polynomial returned by *minAssChar* after modular calculations, the more are chances that the reconstruction yields the true polynomial. Thus, to find the right component of $V(I)$ we pick up from *G* the second, the third and the fourth polynomials, add it to the ideal *I* and compute the decomposition of the obtained ideal with *minAssChar* over the field of rational numbers ($m = 0$). Now, the computation yields component 8 of the theorem (similarly we found component 7). Another possible way to get true components is recomputing with different characteristics. For example, the computation with $m = 139\,907$ gives the correct expression for component 8. One more empirical observation is that, in order to speed up the calculations it is useful first to compute a Groebner basis of the ideal with respect to the degree reverse lexicographic term order, when $minAssGTZ$ is applied, and with respect to the lexicographic term order, when decomposing with *minAssChar*.

Let $J_1, J_2, \ldots, J_8$ be the ideals defining the components $1, 2, \ldots, 8$, respectively. It is easy to check (using the radical membership test or just direct substitutions) that if any of conditions 1–8 is fulfilled then all polynomials $i_4, j_4, \ldots, i_{14}, j_{14}, a_6 - 1, b_6 - 1$ vanish, that is $V_J := \cup_{s=1}^{8} V(J_s) \subseteq V(I)$. We need now to check the opposite inclusion

$$V(I) \subseteq V_J. \tag{23}$$

Since $\cup_{s=1}^{8} V(J_s) = V\left(\cap_{s=1}^{8} J_s\right)$, to verify (23) it is sufficient to show using the radical membership test that any polynomial from $J_{\text{int}} = \cap_{s=1}^{8} J_s$ vanishes on the variety of $V(I)$. We computed J_{int} with the routine *intersect* of *Singular*. Then we have found that all Groebner bases required in the radical membership test are $\{1\}$, but with *Singular* we were able to complete the computations only over the field of characteristic 32 003, so we still could not be sure whether (23) holds for the field of rational numbers. Fortunately, recently, a very efficient package for Groebner bases computation (called FGb) has been developed by Faugère [12]. With this package we have performed the computations over $\mathbb{Q}$ and have checked that the radical membership test returns $\{1\}$ also over this field. Therefore, (23) holds over $\mathbb{Q}$, yielding that equations (1)–(8) of the statement of the theorem define the irreducible decomposition of the variety of the ideal I and, thus, they give the necessary conditions for linearizability of system (21).

(ii) We now prove that the conditions of the theorem are also the sufficient conditions for the linearizability, that is, if the coefficients of system (21) satisfy any of conditions (1)–(8) of the theorem then the system is linearizable.

It is shown in the proof of lemma 2 in [20] that the systems corresponding to the first four cases can be transformed to

$$\begin{aligned} \dot{x} &= x - \tfrac{5}{12}x^5 - \tfrac{7}{6}x^4y + \tfrac{1}{6}x^2y^3 + \tfrac{5}{12}xy^4 + y^5, \\ \dot{y} &= -y - x^5 - \tfrac{5}{12}x^4y - \tfrac{1}{6}x^3y^2 + \tfrac{7}{6}xy^4 + \tfrac{5}{12}y^5 \end{aligned} \tag{24}$$

and the remaining ones to

$$\begin{aligned} \dot{x} &= x - \tfrac{17}{12}x^5 + \tfrac{5}{2}x^4y - \tfrac{5}{6}x^2y^3 - \tfrac{5}{4}xy^4 + y^5, \\ \dot{y} &= -y - x^5 + \tfrac{5}{4}x^4y + \tfrac{5}{6}x^3y^2 - \tfrac{5}{2}xy^4 + \tfrac{17}{12}y^5. \end{aligned} \tag{25}$$

The real system corresponding to (24) is system g of [5]:

$$\dot{u} = -v + \frac{4u^2v}{3}(9u^2 - 8v^2), \qquad \dot{v} = u + \frac{4uv^2}{3}(-3u^2 + 4v^2). \tag{26}$$

It is known (see, e.g., [1, 15]) that the centre at the origin of (26) is isochronous if there is a transversal commuting system. Thus, to prove that (26) is linearizable we show that it has a transversal commuting system of the form

$$\dot{u} = u(1 + 16u^2v^2)H(u, v), \qquad \dot{v} = v\left(1 - \tfrac{16}{3}u^2v^2\right)H(u, v), \tag{27}$$

where

$$H(u,v) = 1 + \sum_{k=1}^{\infty} f_{2k}(v)u^{2k}, \qquad f_2(v) = -\frac{16}{3}v^2, \qquad f_4(v) = \frac{64}{3}v^4 - \frac{20}{3}$$

and $f_{2k}(v)$ are some polynomials. Indeed, it is easy to check that (27) is a commuting system for (26) if and only if

$$(3v - 36u^4v + 32u^2v^3)H'_u + (-3u + 12u^3v^2 - 16uv^4)H'_v + (48u^3v + 32uv^3)H = 0.$$

It yields the recurrence relation for the polynomials f_{2k}:

$$6(k+1)vf_{2k+2}(v) + 32(2k+1)v^3 f_{2k}(v) - (16v^4+3)f'_{2k}(v) + 24(5-3k)vf_{2k-2}(v) + 12v^2 f'_{2k-2}(v) = 0. \tag{28}$$

Noting that f_{2k} defined by (28) are polynomials of v^2 we see that for all $k > 1$ f_{2k+2} can be computed recursively using (28). Therefore, system (26) is isochronous and, hence, system (24) is linearizable[7].

For system (25) there are three invariant curves: $\ell_1 = x - \frac{1}{6}(x-y)^5$, $\ell_2 = y + \frac{1}{6}(x-y)^5$, $\ell_3 = 1 - \frac{5}{12}(x-y)^4$, which yield the Darboux linearization $z_1 = \ell_1\ell_3^{-5/4}$, $z_2 = \ell_2\ell_3^{-5/4}$.

Another linearizing substitution for the corresponding real system

$$\dot{u} = -v + \tfrac{1}{3}(-100u^2 + 16v^2), \qquad \dot{v} = u - \tfrac{20}{3}uv^4$$

is obtained in [5]. □

Theorem 2. *System (21) with $a_6 = b_6 = 0$ is linearizable if and only if $a_3 = b_3 = 0$ and one of the following conditions holds:*

(1) $b_1 = b_4 = a_5 = a_4 = a_2 = 0$,
(2) $b_1 = b_2 = a_5 = a_4 = 0$,
(3) $b_2 = b_4 = b_5 = a_4 = a_1 = 0$,
(4) $b_2 = b_5 = 3a_5 - b_1 = a_4 = 3a_2 - 5b_4 = a_1 = 0$,
(5) $b_4 = a_5 - b_1 = a_4 = a_1 - b_5 = a_2b_2 - 4b_5b_1 = 0$,
(6) $b_1 = b_4 = a_5 = 5a_4 - 3b_2 = a_2 = a_1 - 3b_5 = 0$,
(7) $b_5 = a_5 = 3a_4 - b_2 = a_2 - 3b_4 = a_1a_4^2 + b_4^2b_1 = 0$,
(8) $b_4 = b_5 = a_2 = a_1 = 0$,
(9) $a_5 - 3b_1 = 4b_2^3 + 729b_4b_1^2 = 4b_4b_2 + 3b_5b_1 = b_5b_2^2 - 243b_4^2b_1 = 324b_4^3 + b_5^2b_2 = 9a_4 + b_2 = a_2 + 9b_4 = 3a_1 - b_5 = 0$,
(10) $a_5 + b_1 = a_4 + b_2 = a_2 + b_4 = a_1 + b_5 = 0$.

Proof. To obtain the necessary conditions for linearizability presented above with *minAssChar*, we found the minimal associate primes of $I = \langle i_4, j_4, \ldots, i_{14}, j_{14}, a_6, b_6\rangle$ (for this case, we were able to complete all the calculations with *Singular* over the field of rational numbers). We now prove that they are the sufficient conditions for linearizability. Linearizability of systems 2, 8 and 9 has been proven in [5, 20]. Case 3 is dual to 1 and case 6 is dual to 4 under the involution (22). We consider the remaining cases.

(1) The corresponding system

$$\dot{x} = x - a_1x^5, \qquad \dot{y} = -y + b_5x^4y + b_2xy^4 \tag{29}$$

has three invariant curves: $l_1 = 1 - a_1x^4$, $l_2 = x$, $l_3 = y$. Using (13) (with $c = 0$), we find that the first equation is linearizable by the change $z_1 = x\ell_1^{-1/4}$. In order to

[7] It was an open problem stated in [5] to prove that system (26) is isochronous. Another proof of isochronicity of this system was obtained by Colin Christopher [7].

obtain the linearization of the second equation we look for the first integral. The Darboux integrating factor obtained according to (11) is $\mu = x^{-4}y^{-4}\ell_1^{(a_1+3b_5)/4a_1}$. Using μ by standard computations we find the first integral of (29)

$$H(x,y) = \frac{1}{3}x^{-3}y^{-3}\ell_1^{\frac{3(a_1-b_5)}{4a_1}} - \frac{1}{2}b_2x^{-2}{}_2F_1\left(-\frac{1}{2}, \frac{a_1+3b_5}{4a_1}; \frac{1}{2}; a_1x^4\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function. Then $\Psi(x,y) = (3H(x,y))^{-1/3}$ is the Lyapunov first integral of (29) yielding the linearization of the second equation of (29) in the form

$$z_2 = x^{-1}\ell_1^{1/4}\Psi(x,y).$$

The linearizations are defined for $a_1 \neq 0$. We failed to find an explicit linearization for the case $a_1 = 0$; however we can prove its existence. To see this, one can check that in this case there is a first integral of the form $\Psi(x,y) = \sum_{k=1}^{\infty} f_k(y)x^k$. Thus, since the first equation of the system is already linear, $\Psi(x,y)/x$ is a linearization of the second equation. Another way to prove that the linearization exists is using the following geometrical argument. The set of all linearizable systems is the zero set of a polynomial system; therefore it is closed in the Zariski topology. In our case, we have $b_1 = b_4 = a_5 = a_4 = a_2 = 0$ but $a_1 \neq 0$. That is, we have the set $V(\langle b_1, b_4, a_5, a_4, a_2\rangle)\setminus V(\langle a_1\rangle)$. Obviously, the Zariski closure of this set is again $V(\langle b_1, b_4, a_5, a_4, a_2\rangle)$, that is, a linearization must exist also for the case $a_1 = 0$.

(2) In this case, the corresponding system is linearizable by the transformation

$$z_1 = x\ell_1^{1/12}\ell_2^{-1/2}, \qquad z_2 = y\ell_1^{-1/4}\ell_2^{1/2},$$

where $\ell_1 = 1 - 2b_4x^3y - 3a_5y^4$ and $\ell_2 = 1 - 2b_4x^3y$.

(3) We assume that $b_2 \neq 0$ (otherwise, we have a subcase of 2 or 3). Then the system has the form

$$\dot{x} = x - a_1x^5 - \frac{4a_1b_1}{b_2}x^4y - b_1xy^4, \qquad \dot{y} = -y + a_1x^4y + b_2xy^4 + b_1y^5. \tag{30}$$

There are two invariant curves:

$$\ell_1 = 1 - a_1x^4 - \frac{4a_1b_1}{b_2}x^3y - \frac{4a_1b_1^2}{b_2^2}x^2y^2, \qquad \ell_2 = 1 - \frac{b_2^2}{4b_1}x^2y^2 - b_2xy^3 - b_1y^4,$$

with the cofactors $k_1 = -4a_1x^4 - \frac{8a_1b_1}{b_2}x^3y$, $k_2 = 2b_2xy^3 + 4b_1y^4$, respectively. Using (11) we find the integrating factor $\mu(x,y) = (xy)^{-2}(\ell_1\ell_2)^{-1}$. Let $z(x,y) = \mu(-y + a_1x^4y + b_2xy^4 + b_1y^5)$, $w(x,y) = \mu\big(x - a_1x^5 - \frac{4a_1b_1}{b_2}x^4y - b_1xy^4\big)$. Then there exists a first integral of the form $\int z(x,y)\,\mathrm{d}x + \phi(y)$. Since μ is an integrating factor, we obtain

$$\phi'(y) = -\frac{\partial}{\partial y}\int z(x,y)\,\mathrm{d}x - w(x,y).$$

Calculations give $\phi'(y) = -\int \frac{\partial}{\partial y}z(x,y)\,\mathrm{d}x - w(x,y) = 0$; hence $\widetilde{\Phi} = \int z(x,y)\,\mathrm{d}x$ is a first integral of (30). Taking the series expansion it is easy to see that $\Phi(x,y) = 1/\widetilde{\Phi}(x,y)$ is a Lyapunov first integral of the form (12).

Using the integral, we can construct the linearizations (14) and (16). Namely, the system is linearized by the substitution

$$z_1 = x^{1/2}y^{-1/2}\Phi^{1/2}\ell_1^{-1/4}\ell_2^{1/4}, \qquad z_2 = x^{-1/2}y^{1/2}\Phi^{1/2}\ell_1^{1/4}\ell_2^{-1/4}.$$

(4) We assume $a_4 \neq 0$; otherwise, it was proved in lemma 1 in [20] that the corresponding system is linearizable. Then the system is of the form

$$\dot{x} = x + \frac{b_1 b_4^2}{a_4^2} x^5 - 3b_4 x^4 y - a_4 x^2 y^3, \qquad \dot{y} = -y + b_4 x^3 y^2 + 3a_4 x y^4 + b_1 y^5. \tag{31}$$

It has the invariant curves $\ell_1 = x$, $\ell_2 = y$, $\ell_3 = 1 + \frac{b_1 b_4^2}{a_4^2} x^4 - 4b_4 x^3 y - 4a_4 x y^3 - b_1 y^4$. Let $g(x, y) = \ell_3 - 1$. Performing the change of variables

$$X = x\ell_3^{-1/8}, \qquad Y = y\ell_3^{-1/8} \tag{32}$$

we obtain from (31) the system

$$\dot{X} = X\,(1 + g(x, y)/2)\,, \qquad \dot{Y} = -Y\,(1 + g(x, y)/2)\,. \tag{33}$$

Substituting $x = X\ell_3^{1/8}$ and $y = Y\ell_3^{1/8}$ into $g(x, y)$, we find $g(x, y) = g(X, Y)\sqrt{\ell_3(x, y)}$, yielding $g(X, Y)^2 = g(x, y)^2/(1 + g(x, y))$.

Let us show that there exists a function $m(X, Y)$ such that $\mathrm{d}m(X, Y)/\mathrm{d}t = g(x, y)$. To this end, we need to solve the equation

$$X\frac{\partial m}{\partial X} - Y\frac{\partial m}{\partial Y} = \frac{g(x, y)}{(1 + g(x, y)/2)} = \frac{g(X, Y)}{\sqrt{1 + g(X, Y)^2/4}}. \tag{34}$$

Note that the right-hand side can be expanded in odd powers of $g(X, Y)$. From the form of $g(X, Y)$ (no X^2Y^2 term) we know that an odd power of $g(X, Y)$ has no term of the form $(XY)^n$ which means that we can solve (34) for $m(X, Y)$ with $m(0, 0) = 0$. Then, the substitution $x_1 = X\,\mathrm{e}^{-\frac{1}{2}m(X,Y)}$, $y_1 = Y\,\mathrm{e}^{\frac{1}{2}m(X,Y)}$ (where X and Y are given by (32)) provides a linearization of system (31).[8]

(5) The Darboux linearization for this case is given by $z_1 = x\ell_1^{-1/4}$, $z_2 = y\ell_1^{-1/4}$, where $\ell_1 = 1 + b_5 x^4 + 2b_4 x^3 y - 2b_2 x y^3 - b_1 y^4$. □

To compute the conditions presented in the statement of the following theorem we used the way described in (i) of the proof of theorem 1. In the proof of the theorem, we prove that under these conditions the corresponding systems indeed are linearizable.

Theorem 3. *System (21) with $a_6 = 1$, $b_6 = 0$ is linearizable if and only if $a_3 = b_3 = 0$ and one of the following conditions holds:*

(1) $b_1 = b_2 = b_5 = a_5 = a_4 = a_2 + 5b_4 = a_1 = 0$,
(2) $b_1 = b_2 = b_4 = a_5 = a_4 = a_2 = a_1 + 5b_5 = 0$,
(3) $b_4 = b_5 = a_2 = a_1 = 0$,
(4) $b_1 = b_5 = a_5 = 3a_4 - b_2 = a_2 - 3b_4 = a_1 = 0$,
(5) $a_2 + 3b_4 = 13a_5 - 15b_1 = 9a_4 + b_2 = 3a_1 - b_5 = 24b_1^2 + 169a_4 = 6b_4b_1 + 13a_1 = 2b_2^2 + 27b_4 = 0$,
(6) $b_1 = b_2 = b_4 = a_5 = a_4 = a_2 = 3a_1 + 5b_5 = 0$.

Proof. (1) Observing that the corresponding system has the invariant curves $\ell_1 = xy - \frac{1}{6}y^6$, $\ell_2 = 1 + \frac{b_4}{40}(320x^3y - 240x^2y^6 + 48xy^{11} - 3y^{16})$, we find the Darboux first integral $\Psi(x, y) = \ell_1\ell_2^{-3/8}$ and the linearization $z_1 = \Psi(x, y)/z_2$, $z_2 = y\ell_2^{-1/16}$.

(2) For this case the invariant curves

$$\ell_1 = xy - \frac{1}{6}y^6, \qquad \ell_2 = 1 + \frac{b_5}{112}(560x^4 - 1120x^3y^5 + 420x^2y^{10} - 60xy^{15} + 3y^{20})$$

[8] Originally the authors were able to prove the linearizability of system (31) only for the case when the system is real. The presented way of linearization was suggested by an anonymous referee of the paper.

allow us to construct the Darboux first integral $\Psi(x, y) = \ell_1 \ell_2^{-3/10}$ and the Darboux linearization $z_1 = \Psi(x, y)/z_2$, $z_2 = y\ell_2^{-1/20}$.

(3) The system is written as

$$\dot{x} = x - a_4 x^2 y^3 - a_5 x y^4 - y^5, \qquad \dot{y} = -y + b_2 x y^4 + b_1 y^5. \tag{35}$$

Though we are unable to find an explicit linearizing transformation for (35), we can prove its existence. To this end, we proceed in the spirit of [13]. Namely, we look for a linearizing substitution for the second equation of the system in the form

$$z_2 = \sum_{k=1}^{\infty} f_k(x) y^k, \tag{36}$$

where $f_k(x)$ $(k = 2, 3, \ldots)$ are some polynomials of degree $k - 1$ and $f_1(x) \equiv 1$. Formula (36) provides the linearization if there are $f_k(x)$'s satisfying the differential equation

$$\begin{aligned} &x f_k'(x) + (1-k) f_k(x) + (k-3) b_2 x f_{k-3}(x) - a_4 x^2 f_{k-3}'(x) \\ &\qquad + (k-4) b_1 f_{k-4}(x) - a_5 x f_{k-4}'(x) - f_{k-5}'(x) = 0, \end{aligned} \tag{37}$$

where $f_n(x) \equiv 0$ when $n \leqslant 0$. A straightforward computation gives polynomials $f_2, \ldots, f_5$. Assume that for $k = 6, \ldots, m$ there are polynomials f_k satisfying (37) and such that $\deg(f_k) = k - 1$. Then, for $k = m + 1$, solving the linear differential equation (37), we obtain

$$f_{m+1}(x) = x^m \left(C + \int x^{-m-1} h_{m-2}\, \mathrm{d}x \right) = C x^m + \tilde{h}_{m-2}(x).$$

Since h_{m-2} is a polynomial of degree $m - 2$, we see that $\deg f_{m+1} = m$. Therefore (36) is a linearization of the second equation of (35).

To prove that there is a linearization for the first equation of (35) it is sufficient to show that a Lyapunov first integral of the system can be found in the form

$$\Psi(x, y) = \sum_{k=1}^{\infty} g_k(x) y^k,$$

where $g_1(x) = x$, $g_2(x) = x^2$ and $g_k(x)$ are some polynomials of degree k. The polynomials $g_k(x)$ should fulfil the linear differential equation

$$\begin{aligned} &x g_k'(x) - k g_k(x) + (k-3) b_2 x g_{k-3}(x) - a_4 x^2 g_{k-3}'(x) \\ &\qquad + (k-4) b_1 g_{k-4}(x) - a_5 x g_{k-4}'(x) - g_{k-5}'(x) = 0. \end{aligned} \tag{38}$$

Similarly as above, we see that for all $k = 1, 2, \ldots$ there are polynomials $g_k(x)$ satisfying (38) and such that $\deg(g_k)$ is at most k. Therefore, the first equation of (35) can be linearized by the change $z_1 = \Psi(x, y)/z_2$.

(4) The corresponding system is

$$\dot{x} = x - 3b_4 x^4 y - a_4 x^2 y^3 - y^5, \qquad \dot{y} = -y + b_4 x^3 y^2 + 3a_4 x y^4. \tag{39}$$

Similarly, as for system (35) it is easy to check that the second equation of (39) can be linearized by the change

$$z_2 = \sum_{k=1}^{\infty} h_k(x) y^k,$$

where $h_k(x)$ is some polynomial of degree at most $3(k - 1)$, $h_1(x) \equiv 1$ and the first equation of (39) can be linearized by the change $z_1 = \Psi(x, y)/z_2$, where

$$\Psi(x, y) = \sum_{k=1}^{\infty} g_k(x) y^k$$

with $g_k(x)$ being some polynomial of degree at most $3k-2$ and $g_1(x) = x$, $g_2(x) = b_4x^4$.

(5) In this case, we can write the system as

$$\dot{x} = x - \frac{256a_5^5}{9375}x^5 - \frac{128a_5^4}{625}x^4y + \frac{8a_5^2}{75}x^2y^3 - a_5xy^4 - y^5,$$

$$\dot{y} = -y + \frac{256a_5^5}{3125}x^4y - \frac{128a_5^4}{1875}x^3y^2 + \frac{24a_5^2}{25}xy^4 + \frac{13a_5}{15}y^5.$$

There are three invariant curves:

$$\ell_1 = 1 - \frac{256a_5^4}{1875}x^3y - \frac{64a_5^3}{125}x^2y^2 - \frac{16a_5^2}{25}xy^3 - \frac{4a_5}{15}y^4,$$

$$\ell_2 = 1 - \frac{256a_5^5}{9375}x^4 - \frac{256a_5^4}{1875}x^3y - \frac{32a_5^3}{125}x^2y^2 - \frac{16a_5^2}{75}xy^3 - \frac{a_5}{15}y^4,$$

$$\ell_3 = x - \frac{128a_5^4}{1875}x^4y - \frac{128a_5^3}{375}x^3y^2 - \frac{16a_5^2}{25}x^2y^3 - \frac{8a_5}{15}xy^4 - \frac{1}{6}y^5$$

yielding the linearization $z_1 = \ell_3\ell_1^{-1}\ell_2^{-1/4}$, $z_2 = y\ell_1^{-1}\ell_2^{3/4}$.

(6) The corresponding system is

$$\dot{x} = x + \frac{5b_5}{3}x^5 - y^5, \qquad \dot{y} = -y + b_5x^4y. \tag{40}$$

First, we prove by induction that the system admits a Lyapunov first integral of the form

$$\Psi(x, y) = \sum_{k=0}^{\infty} f_k(x)y^{5k+1},$$

where $f_k(x) = h_{3k+1}(x)(3 + 5b_5x^4)^{-(2k+2/5)}$ with $h_1(x) = 3^{2/5}x$ and $h_{3k+1}(x)$ being some polynomials of degree at most $3k+1$. The functions $f_k(x)$ should satisfy the differential equation

$$\left(x + \tfrac{5}{3}b_5x^5\right) f_k'(x) + (5k+1)(b_5x^4 - 1) f_k(x) - f_{k-1}'(x) = 0. \tag{41}$$

Assume that for $k = 1, \ldots, m$ equation (41) has solutions of the form (41). When $k = m+1$ by solving (41) we obtain

$$\begin{aligned} f_{m+1}(x) &= \frac{x^{5m+6}}{(3+5b_5x^4)^{2m+12/5}} \cdot \int \frac{(3+5b_5x^4)^{2m+7/5}}{x^{5m+7}} (f_m(x))' \,\mathrm{d}x \\ &= \frac{x^{5m+6}}{(3+5b_5x^4)^{2m+12/5}} \cdot \int \frac{(3+5b_5x^4)^{2m+7/5}}{x^{5m+7}} \frac{\tilde{h}_{3m+4}(x)}{(3+5b_5x^4)^{2m+7/5}} \,\mathrm{d}x \\ &= \frac{h_{3(m+1)+1}}{(3+5b_5x^4)^{2(m+1)+2/5}}, \end{aligned}$$

where $\tilde{h}_{3m+4}(x)$ stands for a polynomial of degree at most $3m+4$.

We obtain a linearization using (20). Let $r = P(x, y)/\Psi_y'(x, y)$. Then the system (40) is linearized by the change $z_1 = \Psi(x, y)/z_2$, $z_2 = yr^{-3/28}$. □

4. Final remarks

The isochronicity problem for the real polynomial system in the form of the linear oscillator $\dot{u} = -v$, $\dot{v} = u$ perturbed by the fifth degree homogeneous polynomials, that is, for the system

$$\begin{aligned} \dot{u} &= -v + \alpha_1u^5 + \alpha_2u^4v + \alpha_3u^3v^2 + \alpha_4u^2v^3 + \alpha_5uv^4 + \alpha_6v^5, \\ \dot{v} &= u + \beta_1u^5 + \beta_2u^4v + \beta_3u^3v^2 + \beta_4u^2v^3 + \beta_5uv^4 + \beta_6v^5, \end{aligned} \tag{42}$$

where $u, v, \alpha_i, \beta_i \in \mathbb{R}$, has been studied in [5]. The authors of [5] have found all time-reversible isochronous systems (42) and pointed out two more isochronous systems in family (42).

From theorems 1 and 2, we have derived the necessary and sufficient conditions for isochronicity of system (42). Comparing our conditions for isochronicity of system (42) with those obtained in [5] we have found only one system not appearing in [5]. Namely, these are systems from condition 10 of theorem 3 satisfying the condition $\text{Im}(a_1 a_4^2) \neq 0$ (such systems are not time reversible).

To summarize, we have obtained the necessary and sufficient conditions for linearizability of system (21). By linear substitutions almost all (the exception is the above-mentioned systems from condition 10 of theorem 3) isochronous systems (42) can be transformed to one of systems studied in [5]. A somewhat surprising result of our study is that most cases of isochronicity of system (42) occur in time-reversible systems.

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